**An Introduction to the Method of Proof by Induction.**

*Here is a link to a few YouTube videos by Eddie Woo on proof by induction:* [*https://www.youtube.com/c/misterwootube/search?query=induction*](https://www.youtube.com/c/misterwootube/search?query=induction)

**Just for Fun: Math Corner**

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What is **Proof by the Principle of Mathematical Induction**? Well, from a first glance at this article, it seems to be a long and dreary subject in mathematics, but proof by induction is actually quite easy and quick to learn. This method of proof usually applies to discrete mathematics and number theory as it provides a quick and easy way to prove summations and divisibility.

 There are four steps to proof by induction: 1. Test, 2, Assume, 3. Prove, and 4. Explain—a way to remember this is through the acronym TAPE. Below are example problems for both summations and number divisibility that explain the method of proof by induction, and at the end are a few take-home problems you can attempt yourself.

1. Prove through induction that the sum of the first $n$ natural numbers equals $\frac{n(n+1)}{2}$. We can also phrase this as, $1 + 2 + 3 + . . . + n = \frac{n(n+1)}{2}$. First, we **test the statement for its first possible value**, in this case being $n = 1$. Thus, the Left Hand Side(now will be referenced as LHS) = 1, and the Right Hand Side(which will now be referenced as RHS) = $\frac{1(1 + 1 )}{2} = 1$. The LHS = RHS, therefore($∴$) the statement is true for $n = 1$. Next, we **need to assume that the statement is true for** $n = k$, i.e., $1 + 2 + 3 + . . . + k = \frac{k(k + 1)}{2}$. Thirdly, we **need to prove the statement true for** $n = k + 1$; i.e., $1 + 2 + 3 + . . . + k + (k + 1)= \frac{(k + 1)(k + 1+ 1)}{2} which equals \frac{(k + 1)( k + 2)}{2}$. Now, we need to incorporate the information assumed in the assumption step into the proving step. Because we already know that $1 + 2 + 3 + . . . + k = \frac{k(k + 1)}{2}$, we can substitute this into the assumption step, making $1 + 2 + 3 + . . . + k + (k + 1)= \frac{k(k + 1)}{2} + (k + 1) by assumption$. Now, we need to put these two fractions on top a common denominator, equaling $\frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$(this last step was done by factoring out the common $k + 1$). If take a look at our initial statement in the prove step, the LHS = RHS, meaning that this is a true statement. Our final step is to formalize the information found in this proof; this is done as such:

**If the statement is true for** $n = k$**, then it is also true for** $n = k + 1$**. However, the statement is true for its first value,** $n = 1$, $∴$ **the statement is true for** $n = 1, 2, 3, . . .$**(all positive integers.** This can shortened into one sentence: ***By the Principal of Mathematical Induction, the statement is true for*** $n = 1, 2, 3, . .. or Z^{+}. $

So, in this last problem we learned how to formally and fully use the method of proof by induction, quite a long process but pretty simple. Let’s take a look at another problem, but this time—as we have a semi-familiarity with proof by induction—in a less formal manner.

1. Prove through induction that the sum of the first $n$ natural cubes equals $\frac{n^{2} (n + 1)^{2}}{4}$. In other words, $1^{3} + 2^{3} + 3^{3} + . . . + n^{3 }=\frac{n^{2} (n + 1)^{2}}{4}. $
2. Test the statement for its first possible value, being $n = 1$. $LHS = 1^{3} = 1$, and $RHS = \frac{1^{2} (1 + 1)^{2}}{4} = 1, RHS = LHS ∴the statement is true$.
3. Assume the statement is true for $n = k$, i.e., $ 1^{3} + 2^{3} + 3^{3} + . . . + k^{3 }=\frac{k^{2} (k + 1)^{2}}{4}. $
4. Prove the statement is true for $n = k + 1$, i.e., $1^{3} + 2^{3} + 3^{3} + . . . + k^{3 }+ \left(k+1\right)^{3}=\frac{\left(k+1\right)^{2} \left(k+1+1\right)^{2}}{4}.$ From here, we can begin working on proving this statement. From step (b), we can assume that the following statement is true: $\frac{k^{2} \left(k+1\right)^{2}}{4} + \left(k+1\right)^{3}=\frac{\left(k+1\right)^{2} \left(k+1+1\right)^{2}}{4} = \frac{\left(k+1\right)^{2} \left(k+2\right)^{2}}{4}$. Then, we place the two fractions on a common denominator, equaling $\frac{k^{2} \left(k+1\right)^{2} + 4\left(k+1\right)^{3}}{4} = \frac{\left(k+1\right)^{2}\left(k^{2}+4k+4\right)}{4} = \frac{\left(k+1\right)^{2}\left(k+2\right)^{2}}{4} . LHS = RHS ∴ the statement is true.$
5. ***By the Principal of Mathematical Induction, the statement is true for*** $n \in Z^{+}$.
6. Let’s take a look at one more problem: **prove through the method of mathematical induction that** $7^{2n} - 1 $**is divisible by 24**. Why don’t we turn this into an equation so that the question’s form is similar to that above; the problem then turns into this, $7^{2n }- 1 = 24p$(it is some number multiplied by 24, the only condition is that $p \in N$).
7. Let’s test the statement for its first possible value, in this case being $n =1$, as $7^{2(1)} - 1 = 49 - 1 = 48$ which is a multiple of 24.
8. Let’s assume that the statement is true for $n = k$, i.e., $7^{2k }- 1 = 24q$, where $q\in Ν$.
9. Now we shall prove that the statement is true for $n = k + 1$, i.e., $7^{2(k + 1)} - 1 = 24L$, where $L \in Ν$. Let’s expand the LHS and prove that it equals the RHS.

LHS = ., $7^{2(k + 1)} - 1 = 7^{2k + 2} – 1$*(while this looks wildly different from the problems above, the same algorithm to solving applies. We need to look at the information assumed in step (b) and apply it to our scenario. In the assumption step, the* $7^{2k}$ *is already present, meaning we need to tweak our prove statement to resemble the assumption step).* $49⋅7^{2k} - 1 = [7^{2k }- 1] + 48∙7^{2k} = 24L + 48∙7^{2k} by assumption. Thus, 24 can be factored out, and the expression now equals 24(L + 2⋅7^{2k}$*. Our expression now satisfies the condition of the proof, as the expression is a multiple of 24.* $L $ *can by any positive whole integer, and the sum of*$ q$ *(which by definition is an element of* $N$*) and* $2⋅7^{2k}$ *is also a whole positive integer, making the entire expression an element of* $N$*. Thus, our proof is now complete and we can complete the final step.*

1. By the ***By the Principal of Mathematical Induction, the statement is true for*** $n \in Ν.$

**Here are a few more problems for you do try by yourself:**

1. Prove through the method of mathematical induction that the sum of the first $n $numbers of the form $a\_{n }= 3n – 2$ equals $\frac{n(3n - 1)}{2}$.
2. Prove that for any value of $n \in Ν$ the equality is true: $1∙3 + 2∙5 + . . . + n(2n + 1) = \frac{n(n + 1)(4n + 5)}{6}$.
3. Prove that for any value of $n \in Ν$ the equality is true: $1 + 6 + 20 + . . . + (2n - 1)2^{n -1} = 3 + 2^{n}(2n - 3)$. Trust the process with this one and don’t give up!
4. Prove through the method of mathematical induction that $n^{3} + 5n $is divisible by 6.