Divisibility Problems and Equations in Whole Numbers

Just for Fun: Math Corner

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In my opinion, divisibility problems are one of the most difficult subjects in mathematics; there are so many versions and types that are sometimes barely reminiscent of each other. However, there are a few tricks to some of these problems–this article covers a few of these tricks, along with tips on how to solve equations in integers.

Alright, let's begin with a slightly easier problem:

Prove that the sum of a square of a whole number and that same number is an even number—meaning that it is divisible by 2.

First, we must create an algebraic expression that represents the statement of the problem. We can do so as such:

$$a^2 + a = 2n$$

where a represents our whole, natural number, and n represents any other expression that results from algebraic working–however, remember that n must also be another whole number. Let's begin working with the left-hand-side:

$$LHS = a^2 + a = a(a+1)$$

Now, *a* can be any number, even or odd; for a divisibility proof, we need to show that the expression is divisible by 2 when it falls under either category(in this case being even or odd). Let's consider a first case where is even, meaning that

Then, by substitution, the expression turns into

$a = 2m$ where m is any whole number.

Divisibility by 2 means that the expression must contain a factor of 2 somewhere, which in this case it does. Therefore, when a is any even number, the sum of a number squared and the number itself is divisible by 2. However, we still need to prove that this same condition holds when a is odd. Thus, let's look at a second case where a is odd, or

$a = 2q + 1$, where q is any whole number

Then, by substitution

$$(2q+1)(2q+1+1) = (2q+1)(2q+2) = 2(q+1)(2q+1).$$

Thus, when *a* is any odd number, the expression is also divisible by 2. So, we have proved the statement of the problem. Now, let's try a problem that is connected to a divisibility problem. For these types of problems, we are asked to solve an equation in whole numbers. Let's take a look at one of these problems:

Solve: $(x-2)(xy+4) = 1$ **in whole numbers.**

Firstly, by the requirements in the statement of the problem, the factors cannot equal rational or irrational numbers; they can only be whole (positive or negative) numbers. So, what are the numbers that multiply into 1? Well, 1, 1, -1, and -1. Let's begin with the first scenario:

$$\begin{cases} x - 2 = 1 \\ xy + 4 = 1 \end{cases}$$

Now, we must solve this system of equations. I am going to do this through substitution.

$$\begin{cases} x = 3 \\ xy + 4 = 1 \end{cases}$$

$$3y = -3$$

$$y = -1$$

Thus our first solution is $(3, -1)$.

Now, we move to our next scenario:

$$\begin{cases} x - 2 = -1 \\ xy + 4 = -1 \end{cases}$$

Once again, I am going to solve this through substitution:

$$\begin{cases} x = 1 \\ xy + 4 = -1 \end{cases}$$

$$y = -5$$

Thus, our second solution is $(1, -5)$. These solutions do not violate the requirements of the problem as each solution is an integer. Let's look at a second example:

$$2x^2 + xy = x + 7$$

My advice in beginning this problem is to factor the variables so that the equation resembles the one found in the last problem. This can be done as such:

$$2x^2 + xy - x = 7$$

$$x(2x + y - 1) = 7$$

Now, we must think which factors multiply into 7(1 and 7, and -1 and -7) and set them equal to the factors in the above equation. However, each of these equations can equal each of these factors. Let me explain further from my working below:

$$\begin{cases} x = 7 \\ (2x + y - 1 = 1 \end{cases}$$

Then by substitution:

$$2(7) + y = 2$$

$$y = -12$$

So, our first solution is $(7, -12)$.

But, we also have this case:

$$\begin{cases} x = 1 \\ 2x + y - 1 = 7 \end{cases}$$

Then by substitution:

$$2 + y = 8$$

$$y = 6$$

Thus our second solution is $(1, 6)$. Now, we find the solutions for the negative factors.

$$\begin{cases} x = -1 \\ 2x + y - 1 = -7 \end{cases}$$

$$2(-1) + y = -6$$

$$y = -4$$

Our third solution is $(-1, -4)$.

Lastly:

$$\begin{cases} x = -7 \\ 2x + y - 1 = -1 \end{cases}$$

$$2(-7) + y = 0$$

$$y = -14$$

And, our last solution is $(-7, -14)$.

Here are some problems for you to solve:

1. Solve $x^2 - xy - 2y^2 = 1$ in integers.
2. Solve $x^2 + xy - 2y^2 - x + y = 3$ in integers.
3. Solve $x^2 - 3xy + 2y^2 = 3$ in integers.